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On the background field method beyond one loop: a manifestly covariant derivative expansion in super Yang-Mills theories

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Abstract: There are currently many string inspired conjectures about the structure of the low-energy effective action for super Yang-Mills theories which require explicit multi-loop calculations. In this paper, we develop a manifestly covariant derivative expansion of superspace heat kernels and present a scheme to evaluate multi-loop contributions to the effective action in the framework of the background field method. The crucial ingredient of the construction is a detailed analysis of the properties of the parallel displacement propagators associated with Yang-Mills supermultiplets in $\mathcal{N}$-extended superspace.

Keywords: Extended Supersymmetry, Supersymmetric Effective Theories, Superspaces.

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1. Introduction and outlook

The background field method [1]–[10] is a powerful tool for the study of quantum dynamics in gauge theories, including gravity. Its main merit is a manifestly gauge invariant definition of the quantum effective action. Renormalization of the Yang-Mills theories in this approach is not much harder to establish [4, 11] than in the conventional one, and the S-matrix is correctly reproduced [12, 13, 14]. The background field method is universal enough to admit natural superspace generalizations; such extensions have been constructed for $\mathcal{N} = 1$ super Yang-Mills theories and supergravity [15]–[19] and for $\mathcal{N} = 2$ super Yang-Mills theories [20] (see also [21] for earlier attempts to develop a background field formulation for some $\mathcal{N} = 2$ SYM models, including the $\mathcal{N} = 4$ SYM theory).

The price to pay for the manifest gauge invariance of the effective action is that one has to deal with propagators in arbitrary background fields, and these are impossible to compute exactly. At the one-loop level, this does not lead to any serious problems, since the famous Schwinger-DeWitt technique [22, 23] and its numerous generalizations [24]–[30] allow one to compute the effective action to any given order in the derivative expansion. Beyond the one-loop approximation, however, the full power of the background field method has never been exploited. A necessary precursor to such multi-loop calculations should be a manifestly gauge covariant derivative expansion of the Feynman propagators in the presence of arbitrary background fields. This has essentially existed for non-supersymmetric theories since the mid 1980's [24, 27] (although, in our opinion, its significance has not yet been fully appreciated), while such an approach has never been elaborated for supersymmetric gauge theories in superspace. In the non-supersymmetric case, a manifestly covariant calculus has only been worked out in detail for computing the UV divergences of the effective action [33].

Otherwise (and often for counterterm calculations only), it has been usual to resort to one of the following options[1] (see [31, 32] for a more detailed discussion):

(i) a combination of conventional perturbation theory with the background field formalism, with manifest background gauge invariance sacrificed at intermediate stages [8, 35, 36, 15, 17, 19, 37, 38];

(ii) the use of normal coordinates in curved space [39, 40] or the Schwinger-Fock gauge for Yang-Mills fields (see [25] and reference therein), usually in conjunction with momentum space methods.

One can hardly overestimate the importance of developing a manifestly gauge covariant derivative expansion of the *finite* part of the (low energy) effective action beyond the one-loop approximation.[2] A great many research groups have contributed to solving the problem in the non-supersymmetric case, and it is simply impossible to give all references here. The present paper is aimed at completing this program for the case of supersymmetric Yang-Mills theories. Crucial for our considerations were the ideas and techniques developed by DeWitt [23] (in particular the concept of parallel displacement propagator), Barvinsky and Vilkovisky [24] (including the covariant Taylor expansion), as well as Avramidi's extension [27] of the covariant calculus introduced in [24]. Of course, in the beginning was Schwinger [22].

This paper is organized as follows. In section 2 we develop a manifestly covariant derivative expansion of the heat kernels in ordinary Yang-Mills theories, extending and generalizing previous results [23, 24, 27]. In section 3 we outline a manifestly covariant scheme for evaluating multi-loop diagrams in Yang-Mills theories in the framework of the background field method. Section 4 is devoted to a detailed analysis of the properties of the parallel displacement propagators associated with Yang-Mills supermultiplets in $\mathcal{N}$-extended flat superspace. A superspace covariant Taylor expansion is also derived. The Fock-Schwinger gauge in $\mathcal{N}$-extended superspace is described in detail for completeness.[3] In section 5 we provide a manifestly covariant derivative expansion of the superfield heat kernels in $\mathcal{N} = 1$ super Yang-Mills theories. The exact superpropagator in a covariantly constant background is computed in section 6. Finally, the appendix A contains the proof of a technical lemma.

To keep the background covariance of the effective action manifest, Barvinsky and Vilkovisky [24], and more recently Börnsen and van de Ven [32], eliminated the momentum representation from their consideration. Our approach differs conceptually from [24, 32] in that we *do* preserve the momentum representation in a manifestly covariant way.

[1] There also exists the so-called string-inspired approach to perturbative QFT, see [34] for a review and references.

[2] In this paper we are interested in the local (or low energy) expansion of the effective action. Concerning the nonlocal part of the effective action, the interested reader is referred to [41] for a recent comprehensive discussion and references.

[3] Unlike numerous previous quantum calculations with background fields, at *no* stage in our scheme is there need to use the Fock-Schwinger gauge.

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The approach developed in this paper opens the way to computing low-energy effective actions in super Yang-Mills theories beyond one loop. A supergravity extension is quite feasible, and old results on the proper-time technique in curved $\mathcal{N}=1$ superspace [42, 43, 44] should be relevant. We believe that the results will also be helpful, for instance, for a better understanding of (i) numerous non-renormalization theorems which are predicted by the AdS/CFT conjecture and relate to the explicit structure of the low energy effective action in $\mathcal{N}=4$ super Yang-Mills theory (see [45, 46] for more details and references); (ii) quantum deformation of the superconformal symmetry in $\mathcal{N}=2,4$ SCFTs [47] at higher loops.

2. Non-supersymmetric case

We consider a Green's function, $G^i{}_{i'}(x,x') = \mathrm{i}\,\langle\varphi^i(x)\,\bar\varphi_{i'}(x')\rangle$, associated with a quantum field φ, which transforms in some representation of the gauge group, and its conjugate $\bar\varphi$. The Green's function satisfies the equation

$$\Delta_x\, G(x,x') = -\delta^d(x-x')\,\mathbf{1}\,, \qquad \Delta = \nabla^m\nabla_m + \mathcal{P}\,, \qquad \mathbf{1} = (\delta^i{}_{i'})\,, \tag{2.1}$$

with $\nabla_m = \partial_m + \mathrm{i}\,A_m$ the gauge covariant derivatives, $[\nabla_m,\nabla_n] = \mathrm{i}\,F_{mn}$, and $\mathcal{P}(x)$ a local matrix function of the background field containing a mass term $(-m^2)\mathbf{1}$. With respect to the gauge group, ∇_m and $\mathcal{P}$ transform as follows

$$\nabla_m \to \mathrm{e}^{\mathrm{i}\lambda(x)}\,\nabla_m\,\mathrm{e}^{-\mathrm{i}\lambda(x)}\,, \qquad \mathcal{P} \to \mathrm{e}^{\mathrm{i}\lambda(x)}\,\mathcal{P}\,\mathrm{e}^{-\mathrm{i}\lambda(x)}\,, \tag{2.2}$$

and therefore

$$G(x,x') \to \mathrm{e}^{\mathrm{i}\lambda(x)}\,G(x,x')\,\mathrm{e}^{-\mathrm{i}\lambda(x')}\,. \tag{2.3}$$

We introduce the proper time representation of G:

$$G(x,x') = \mathrm{i}\int_0^\infty \mathrm{d}s\,K(x,x'|s)\,, \tag{2.4}$$

where the so-called heat kernel $K(x,x'|s)$ is formally given by

$$K(x,x'|s) = \mathrm{e}^{\mathrm{i}s\,(\Delta+\mathrm{i}\,\varepsilon)}\,\delta^d(x-x')\,\mathbf{1}\,, \qquad \varepsilon\to+0\,, \tag{2.5}$$

and which transforms as

$$K(x,x'|s) \to \mathrm{e}^{\mathrm{i}\lambda(x)}\,K(x,x'|s)\,\mathrm{e}^{-\mathrm{i}\lambda(x')} \tag{2.6}$$

with respect to the gauge group.

It is advantageous to make use of the Fourier integral representation

$$\delta^d(x-x') = \int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\,k.(x-x')}$$

for the delta-function in (2.5). To preserve the gauge transformation law (2.6), however, we should actually represent the full delta-function $\delta^d(x-x')\,\mathbf{1}$ as follows:

$$\delta^d(x-x')\,\mathbf{1} = \int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,\mathcal{I}(x,x')\,. \tag{2.7}$$

Here the matrix $\mathcal{I}(x,x')$ is a functional of the background field chosen in such a way that

(i) it possesses the gauge transformation law

$$\mathcal{I}(x,x') \to \mathrm{e}^{\mathrm{i}\lambda(x)}\,\mathcal{I}(x,x')\,\mathrm{e}^{-\mathrm{i}\lambda(x')}\,; \tag{2.8}$$

(ii) it satisfies the boundary condition

$$\mathcal{I}(x,x) = \mathbf{1}\,. \tag{2.9}$$

As a result, the heat kernel takes the form

$$K(x,x'|s) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,\mathrm{e}^{\mathrm{i}s[(\nabla+\mathrm{i}k)^2+\mathcal{P}]}\,\mathcal{I}(x,x') \equiv \hat{K}(x,x'|s)\,\mathcal{I}(x,x')\,. \tag{2.10}$$

It is clear that the gauge transformation law of the operator $\hat{K}(x,x'|s)$ is

$$\hat{K}(x,x'|s) \to \mathrm{e}^{\mathrm{i}\lambda(x)}\,\hat{K}(x,x'|s)\,\mathrm{e}^{-\mathrm{i}\lambda(x)}\,. \tag{2.11}$$

Given a *gauge invariant* scalar field $\Upsilon(x)$ of compact support, one can prove the following operator identity:

$$\hat{K}(x,x'|s)\cdot\hat{\Upsilon}(x) = \hat{\Upsilon}(x')\cdot\hat{K}(x,x'|s)\,, \qquad \hat{\Upsilon}(x) \equiv \Upsilon(x)\,\mathbf{1}\,, \tag{2.12}$$

see section 5 for a proof. This identity implies

$$\hat{K}(x,x'|s) = \mathbf{:}\,\mathrm{e}^{(x'-x).\nabla}\mathbf{:}\,\hat{A}(x,x'|s)\,, \tag{2.13}$$

where the matrix $\hat{A}(x,x'|s)$ is a functional of the background field such that

$$\hat{A}(x,x'|s)\cdot\hat{\Upsilon}(x) = \hat{\Upsilon}(x)\cdot\hat{A}(x,x'|s)\,. \tag{2.14}$$

In other words, the matrix $\hat{A}(x,x'|s)$ may depend on the covariant derivatives only via multiple commutators starting with the master commutators $[\nabla_m,\nabla_n] = \mathrm{i}\,F_{mn}$ and $[\nabla_m,\mathcal{P}] = (\nabla_m\mathcal{P})$. The operator $\mathbf{:}\,\mathrm{e}^{(x'-x).\nabla}\mathbf{:}$ is defined by

$$\mathbf{:}\,\mathrm{e}^{(x'-x).\nabla}\mathbf{:} = \sum_{p=0}^{\infty}\frac{1}{p!}(x'-x)^{m_1}\cdots(x'-x)^{m_p}\,\nabla_{m_1}\cdots\nabla_{m_p} \tag{2.15}$$

when acting on a gauge covariant field $\varphi(x)$ possessing a covariant Taylor series (see below). More generally, the operator $\mathbf{:}\,\mathrm{e}^{(x'-x).\nabla}\mathbf{:}$ is defined by

$$\mathbf{:}\,\mathrm{e}^{(x'-x).\nabla}\mathbf{:}\,\varphi(x) = I(x,x')\,\varphi(x')\,, \tag{2.16}$$

where $I(x,x')$ is the parallel displacement propagator (see below).

It follows from (2.12) that in the coincidence limit, $\hat{K}(x,x|s)$ is not a differential operator,

$$\hat{K}(x,x|s)\cdot\hat{\Upsilon}(x) = \hat{\Upsilon}(x)\cdot\hat{K}(x,x|s)\,. \tag{2.17}$$

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The latter observation can be restated as the fact that all covariant derivatives in

$$\hat{K}(x,x|s) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \exp\left\{\mathrm{i}s\left[(\nabla + \mathrm{i}k)^2 + \mathcal{P}\right]\right\} \tag{2.18}$$

can be organized into multiple commutators upon doing the (gaussian) momentum integration.

It is well known that the one-loop effective action can be expressed via the functional trace of the heat kernel,

$$\begin{aligned} \operatorname{Tr} K(s) &= \int \mathrm{d}^d x \operatorname{tr} K(x,x|s)\,, \\ K(x,x|s) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\exp\left\{\mathrm{i}s\left[(\nabla + \mathrm{i}k)^2 + \mathcal{P}\right]\right\} \mathcal{I}(x,x')\right)\Big|_{x'=x} \end{aligned} \tag{2.19}$$

where 'tr' denotes the trace over gauge group indices. In accordance with the above discussion, $\hat{K}(x,x|s)$ is not a differential operator, and we then get

$$K(x,x|s) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \exp\left\{\mathrm{i}s\left[(\nabla + \mathrm{i}k)^2 + \mathcal{P}\right]\right\}\,, \tag{2.20}$$

where the boundary condition (2.9) has been taken into account. We therefore conclude that, apart from the structural requirements (2.8) and (2.9), the concrete choice of $\mathcal{I}(x,x')$ is not relevant at the one-loop level.

As already noted, the operator $\hat{A}(x,x'|s)$ depends on the gauge covariant derivatives only via commutators. In practice, it is quite nontrivial to manifestly organize the terms in the expansion of $\hat{A}$ into such commutators. It turns out, however, that such an organization occurs automatically for a special choice of $\mathcal{I}(x,x')$, to be discussed below.

Beyond the one-loop approximation, we should work with the Feynman propagator $G(x,x')$ at $x \neq x'$. Therefore, it would be desirable to choose $\mathcal{I}(x,x')$ in such a way that the heat kernel (2.10) takes a simple form. The best choice turns out to be the so-called parallel displacement propagator $I(x,x')$ along the geodesic connecting the points x' and x [23]. This object (which generalizes the Schwinger phase factor [22, 48]) satisfies the equation[4]

$$(x-x')^a\, \nabla_a\, I(x,x') = 0\,, \tag{2.21}$$

which implies

$$(x'-x)^{a_1}\ldots(x'-x)^{a_n}\, \nabla_{a_1}\ldots\nabla_{a_n}\, I(x,x') = 0\,, \tag{2.22}$$

for any positive integer n. The latter means that the operator $:\mathrm{e}^{(x'-x)\cdot\nabla}:$ in (2.13) collapses to the unit operator when acting on $I(x,x')$. It is worth noting that the identities (2.22) lead to

$$\nabla_{(a_1}\ldots\nabla_{a_n)}\, I(x,x')\Big|_{x=x'} = 0\,, \tag{2.23}$$

for any positive integer n.

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[4]The unique solution to the equation (2.21) under the boundary condition (2.9) is given by $I(x,x') = \mathrm{P}\exp\left(-\mathrm{i}\int_{x'}^{x} A_m\, \mathrm{d}x^m\right)$, where the integration is carried out along the straight line connecting the points x' and x. This explicit form of $I(x,x')$ is not required for actual loop calculations.

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In what follows, we identify $\mathcal{I}(x,x')$ with the parallel displacement propagator.[5] From the equation (2.21), the transformation law (2.8) and the boundary condition (2.9), one can easily deduce [24]

$$I(x,x')\, I(x',x) = \mathbf{1}\,. \tag{2.24}$$

Under hermitean conjugation, this object transforms as

$$\Big(I(x,x')\Big)^{\dagger} = I(x',x)\,. \tag{2.25}$$

Let $\varphi(x)$ be a field transforming in some representation of the gauge group. It can be expanded in a *covariant Taylor series* [24]

$$\varphi(x) = I(x,x') \sum_{n=0}^{\infty} \frac{1}{n!}\,(x-x')^{a_1}\dots(x-x')^{a_n}\,\nabla_{a_1}\dots\nabla_{a_n}\varphi(y)\Big|_{y=x'}\,, \tag{2.26}$$

see section 4 for a proof. We can apply this expansion to $\nabla_b I(x,x')$ considered as a function of x:

$$\nabla_b I(x,x') = I(x,x') \sum_{n=0}^{\infty} \frac{1}{n!}\,(x-x')^{a_1}\dots(x-x')^{a_n}\,\nabla_{a_1}\dots\nabla_{a_n}\nabla_b I(y,x')\Big|_{y=x'}\,. \tag{2.27}$$

Using the identity

$$\nabla_{(a_1}\dots\nabla_{a_n}\nabla_{b)} I(y,x')\Big|_{y=x'} = 0$$

in conjunction with the commutation relation $[\nabla_m,\nabla_n] = \mathrm{i}\,F_{mn}$, one can express

$$\nabla_{(a_1}\dots\nabla_{a_n)}\nabla_b I(y,x')\Big|_{y=x'}$$

in terms of covariant derivatives of the field strength (see the appendix A for more detail). One ends up with [27]

$$\begin{aligned}\nabla_b I(x,x') \,&= \mathrm{i}\, I(x,x') \sum_{n=1}^{\infty} \frac{n}{(n+1)!}\,(x-x')^{a_1}\dots(x-x')^{a_{n-1}}(x-x')^{a_n} \times \\ &\quad \times \nabla'_{a_1}\dots\nabla'_{a_{n-1}} F_{a_n\, b}(x')\,.\end{aligned} \tag{2.28}$$

This relation has a fundamental significance in our considerations. It expresses covariant derivatives of $I(x,x')$ in terms of $I(x,x')$ itself and covariant derivatives of the field strength F_{ab}. This is the property which automatically results in the organization of the covariant derivatives in $\hat{A}(x,x'|s)$ into commutators.

Relation (2.28) can be rewritten in an integral form

$$\begin{aligned}\nabla_b I(x,x') \,&= \mathrm{i}\int_0^1 \mathrm{d}t\, I(x,x(t))\, \frac{\partial x^d(t)}{\partial x^b}\,\dot{x}^c(t)\, F_{cd}(x(t))\, I(x(t),x')\,, \\ x(t) \,&= (x-x')\,t + x'\,,\end{aligned} \tag{2.29}$$

[5] Representation (2.10) with $\mathcal{I}(x,x')$ chosen to be the parallel displacement propagator, was the starting point of Avramidi's analysis [27].

which was derived, in a more general setting, many years ago in [49]. Applying the covariant Taylor expansion to $I(x,x(t))\,F_{cd}(x(t))$,

$$I(x,x(t))\,F_{cd}(x(t))\,I(x(t),x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,(1-t)^n\,(x-x')^{a_1}\dots(x-x')^{a_n}\,\nabla_{a_1}\dots\nabla_{a_n}F_{cd}(x)\,, \tag{2.30}$$

and making use of the identity

$$I(x,x(t))\,I(x(t),x') = I(x,x')\,, \tag{2.31}$$

eq. (2.29) leads to

$$\begin{aligned}\nabla_b I(x,x') \;&=\; -\mathrm{i}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\,(x-x')^{a_1}\dots(x-x')^{a_{n-1}}(x-x')^{a_n}\\ &\quad\times\nabla_{a_1}\dots\nabla_{a_{n-1}}F_{a_n\,b}(x)\,I(x,x')\,.\end{aligned} \tag{2.32}$$

Of course, the latter relation can be deduced directly from (2.28).

It is worth noting that the relation (2.28) and the delta function representation

$$\delta^d(x-x')\,\mathbf{1} = \int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,I(x,x') \tag{2.33}$$

make quite obvious the standard property of the delta-function,

$$\nabla_a\left(\delta^d(x-x')\,\delta^i{}_{i'}\right) = -\nabla'_a\left(\delta^d(x-x')\,\delta^i{}_{i'}\right). \tag{2.34}$$

Relations (2.28) and (2.32) simplify drastically in the case of a covariantly constant gauge field,

$$\nabla_a F_{bc} = 0\,. \tag{2.35}$$

Then, eqs. (2.28) and (2.32) take the form

$$\nabla_b I(x,x') = \frac{\mathrm{i}}{2}\,(x-x')^a\,I(x,x')\,F_{ab}(x') = \frac{\mathrm{i}}{2}\,(x-x')^a\,F_{ab}(x)\,I(x,x')\,. \tag{2.36}$$

Let us fix some space-time point x' and consider the following gauge transformation:

$$\mathrm{e}^{\mathrm{i}\lambda(x)} = I(x',x)\,,\qquad \mathrm{e}^{\mathrm{i}\lambda(x')} = \mathbf{1}\,. \tag{2.37}$$

Applying this gauge transformation to $I(x,x')$, as in eq. (2.8), the result is

$$I(x,x') = \mathbf{1}\,, \tag{2.38}$$

which is equivalent, due to (2.21), to the Fock-Schwinger gauge [50, 22]

$$(x-x')^m\,A_m(x) = 0\,. \tag{2.39}$$

In the Fock-Schwinger gauge, the relation (2.28) becomes [51]

$$A_b(x) = \sum_{n=1}^{\infty}\frac{n}{(n+1)!}\,(x-x')^{a_1}\dots(x-x')^{a_{n-1}}(x-x')^{a_n}\,\nabla'_{a_1}\dots\nabla'_{a_{n-1}}F_{a_n\,b}(x')\,. \tag{2.40}$$

Thus, all coefficients in the Taylor expansion of $A(x)$ acquire a geometric meaning.

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We now comment on the explicit evaluation of the kernel

$$K(x,x'|s) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\,k.(x-x')}\, \mathrm{e}^{\mathrm{i}s[(\nabla+\mathrm{i}k)^2+\mathcal{P}]}\, I(x,x')\,. \tag{2.41}$$

It should be pointed out that we are interested in a manifestly covariant expansion of the heat kernel which can be truncated at required order in the derivative expansion. Rescaling k, the right hand side of (2.41) is the result of applying the operator

$$\frac{1}{(4\pi^2 s)^{d/2}} \int \mathrm{d}^d k\, \mathrm{e}^{-\mathrm{i}k^2+\mathrm{i}\,s^{-1/2}\,k.(x-x')}\, \mathrm{e}^{[\mathrm{i}s\nabla^2-2s^{1/2}k.\nabla+\mathrm{i}\,s\mathcal{P}]} \tag{2.42}$$

to the parallel displacement propagator. The second exponential factor here should then be expanded in a Taylor series. Whenever a covariant derivative ∇_b from this series hits $I(x,x')$, we apply the relation (2.28). Given a product of the form $\mathcal{P}(x)\,I(x,x')$, we represent it as

$$\mathcal{P}(x)\,I(x,x') = I(x,x') \sum_{n=0}^{\infty} \frac{1}{n!}\,(x-x')^{a_1}\ldots(x-x')^{a_n}\, \nabla'_{a_1}\ldots\nabla'_{a_n}\mathcal{P}(x')\,. \tag{2.43}$$

A generic term in the Taylor expansion will involve a gaussian moment of the form

$$\langle k^{a_1}\ldots k^{a_n}\rangle \equiv \frac{1}{(4\pi^2 s)^{d/2}} \int \mathrm{d}^d k\, \mathrm{e}^{-\mathrm{i}k^2+\mathrm{i}\,s^{-1/2}\,k.(x-x')}\, s^{1/2}k^{a_1}\ldots s^{1/2}k^{a_n}\,, \tag{2.44}$$

where each k^{a_i} comes together with an s-independent factor of ∇_{a_i}; there also occur insertions of $s\nabla^2$ and $s\mathcal{P}$. To compute the moments (2.44), we can introduce a generating function $Z(J)$,

$$\begin{aligned} Z(J) &= \frac{1}{(4\pi^2 s)^{d/2}} \int \mathrm{d}^d k\, \mathrm{e}^{-\mathrm{i}k^2+\mathrm{i}\,s^{-1/2}\,k.(x-x')\,+s^{1/2}\,J.k}\,, \\ \langle k^{a_1}\ldots k^{a_n}\rangle &= \frac{\partial^n}{\partial J_{a_1}\ldots\partial J_{a_n}}\, Z(J)\Big|_{J=0}\,. \end{aligned} \tag{2.45}$$

One readily gets

$$Z(J) = \frac{\mathrm{i}}{(4\pi\mathrm{i}s)^{d/2}}\, \mathrm{e}^{\mathrm{i}(x-x')^2/4s}\, \mathrm{e}^{-\mathrm{i}sJ^2/4+J.(x-x')/2}\,. \tag{2.46}$$

Then, the resulting expression for the kernel is of the form

$$K(x,x'|s) = \frac{\mathrm{i}}{(4\pi\mathrm{i}s)^{d/2}}\, \mathrm{e}^{\mathrm{i}(x-x')^2/4s}\, F(x,x'|s)\,, \qquad F(x,x'|s) = \sum_{n=0}^{\infty} a_n(x,x')\,(\mathrm{i}s)^n\,, \tag{2.47}$$

where[6]

$$a_0(x,x') = \boldsymbol{:}\mathrm{e}^{(x'-x).\nabla}\boldsymbol{:}\, I(x,x') = I(x,x')\,. \tag{2.48}$$

[6]Had one started with the more general representation (2.10) instead of (2.41), the result for a_0 would then have been $a_0(x,x') = \boldsymbol{:}\mathrm{e}^{(x'-x).\nabla}\boldsymbol{:}\mathcal{I}(x,x') = I(x,x')$.

This is consistent with the standard Schwinger-DeWitt asymptotic expansion of the heat kernel [23]. By construction, the coefficients a_n have the form

$$
\begin{aligned}
a_n(x,x') &= \mathbf{a}_n\left(F(x),\nabla F(x),\ldots,\mathcal{P}(x),\nabla\mathcal{P}(x)\ldots;x-x'\right) I(x,x') \\
&= I(x,x')\,\mathbf{a}'_n\left(F(x'),\nabla' F(x'),\ldots,\mathcal{P}(x'),\nabla'\mathcal{P}(x')\ldots;x-x'\right), \qquad (2.49)
\end{aligned}
$$

where the functions $\mathbf{a}_n$ and $\mathbf{a}'_n$ are straightforward to compute using the scheme described above. In the standard Schwinger-DeWitt technique, one has to solve the recurrence relations which follow from the equation

$$
\left(\mathrm{i}\,\frac{\partial}{\partial s}+\nabla^2+\mathcal{P}\right) K(x,x'|s)=0\,. \qquad (2.50)
$$

In practice, these recurrence relations prove more difficult to solve than to implement the above perturbation scheme.

3. Evaluating multi-loop diagrams

In section 2, a covariant procedure for the derivative expansion of Green's functions in the background field method was presented. We now extend this to the evaluation of Feynman diagrams.

As already discussed in section 2, at the one loop level, the effective action can be expressed in terms of the functional trace of the heat kernel, and therefore only a knowledge of the coincidence limit of the kernel is required. It was noted that in principle this does not require a specific choice of $\mathcal{I}(x,x')$ in (2.19) because, after doing the momentum integration, all covariant derivatives assemble into commutators and there are actually no derivatives of $\mathcal{I}(x,x')$ to be evaluated. In practice, it is quite difficult to assemble the commutators, and it proves efficient to use the procedure outlined in section 2: namely, to choose $\mathcal{I}(x,x')$ to be the parallel displacement propagator $I(x,x')$ and to evaluate the action of covariant derivatives on $I(x,x')$ using either (2.28) or (2.32). This automatically generates the required commutators.

Beyond one loop, the situation is more complicated, in that diagrams involve multiple Green's functions, and interaction vertices will in general include insertions of background fields and covariant derivatives. When covariant derivatives act on Green's functions, the generic structure will be of the form

$$
\begin{aligned}
\nabla_{a_1}\cdots\nabla_{a_n}\nabla'_{b_1}\cdots\nabla'_{b_m}K(x,x'|s) = \int\frac{\mathrm{d}^d k}{(2\pi)^d}\,&\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,(\nabla+\mathrm{i}k)_{a_1}\cdots(\nabla+\mathrm{i}k)_{a_n}\times \\
&\times\mathrm{e}^{\mathrm{i}s[(\nabla+\mathrm{i}k)^2+\mathcal{P}]}(\nabla'-\mathrm{i}k)_{b_1}\cdots(\nabla'-\mathrm{i}k)_{b_m}I(x,x')\,, \quad (3.1)
\end{aligned}
$$

where the proper time integral has been omitted. When ∇ hits $I(x,x')$, the result can be represented as either (2.28) or (2.32). When ∇' hits $I(x,x')$, the result can be evaluated using

$$
\nabla'_b\,I(x,x') = -I(x,x')\,\nabla'_b I(x',x)\,I(x,x')\,. \qquad (3.2)
$$

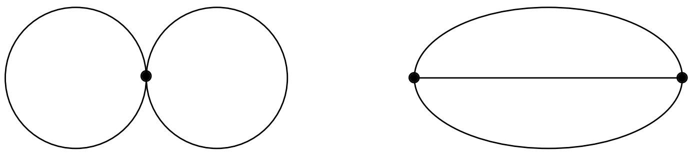

Figure 1: Two-loop graphs: 'eight' diagram and 'fish' diagram.

After having applied all the covariant derivatives to $I(x,x')$, the integrand in (3.1) can be expressed either as

$$\mathrm{e}^{\mathrm{i}\,[k.(x-x')-s\,k^2]}\,\Psi\,\big(F(x),\nabla F(x),\ldots,\mathcal{P}(x),\nabla\mathcal{P}(x),\ldots;x-x',k,s\big)\,I(x,x') \qquad (3.3)$$

or

$$\mathrm{e}^{\mathrm{i}\,[k.(x-x')-s\,k^2]}\,I(x,x')\,\Psi'\,\big(F(x'),\nabla' F(x'),\ldots,\mathcal{P}(x'),\nabla'\mathcal{P}(x'),\ldots;x-x',k,s\big)\,. \qquad (3.4)$$

At this stage, there are two possible ways to proceed. For any Feynman graph, the momentum integrals in *all* kernels (3.1) can either be

(i) left to the end of calculation; or

(ii) carried out first.

These choices give rise to two different *manifestly* covariant perturbation schemes:

(i) momentum space scheme;

(ii) configuration space scheme.

In our opinion, the former is most suitable for arbitrary backgrounds, while the latter is better adapted to special background configurations such as a covariantly constant field. We will describe in detail the momentum space scheme. It is worth noting that the configuration space scheme was advocated in [24, 31, 32].

For the purposes of illustration, we first consider two-loop contributions to the effective action. The relevant Feynman diagrams are either of the 'eight' type or 'fish' type, as shown in figure 1.

The 'eight' diagram is relatively simple to treat since it involves the product of two Green's functions or their covariant derivatives in a coincidence limit. The situation for the 'fish' diagram is somewhat more complex. This diagram makes a contribution to the effective action of the form (with gauge indices suppressed)

$$\int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \int_0^\infty \mathrm{d}s_3 \int \mathrm{d}^d x \int \mathrm{d}^d x'\, V_1(x)\,V_2(x')\,K_1(x,x'|s_1)\,K_2(x,x'|s_2)\,K_3(x,x'|s_3)\,, \quad (3.5)$$

where the vertex factors $V_1(x)$ and $V_2(x')$ will in general contribute insertions of the background fields, and may also contain covariant derivatives which act on one or more of the kernels. The procedure for evaluating (3.5), to be outlined below, is to reduce this diagram to a 'skeleton'. The latter is characterized by the following properties:

(i) all explicit background field dependence is collected at a single vertex, say x';

(ii) the vertex at x is connected to that at x' by four (undifferentiated) parallel displacement propagators only;

(iii) the skeleton is gauge invariant at each vertex.

The kernels and their derivatives in (3.5) should be expressed as in (3.1) and (3.4). In particular, the background field dependence in the kernels coming from either $\mathcal{P}$ or from covariant derivatives of $I(x,x')$ are evaluated at x', with $\mathcal{P}(x)\, I(x,x')$ treated as in (2.43). The x-dependence of each of the kernels is then only via a multiplicative factor $I(x,x')$, Taylor factors $(x-x')^n$, and via a phase factor $\mathrm{e}^{\mathrm{i}k_i\cdot(x-x')}$, where k_i ($i = 1, 2$ or 3) denotes the momentum associated with the kernel. In contrast, the vertices $V_1(x)$ and $V_2(x')$ contribute background fields evaluated at x and x' respectively. As stated earlier, the aim is to get all explicit dependence on the background to be in the form of fields evaluated at x'. This can be achieved by the covariant Taylor expansion of the product of fields in $V_1(x)$ in terms of fields evaluated at x',

$$\mathcal{V}_1(x) = I(x,x') \sum_{n=0}^{\infty} \frac{1}{n!}\, (x-x')^{a_1} \ldots (x-x')^{a_n}\, \nabla'_{a_1} \ldots \nabla'_{a_n} \mathcal{V}_1(x')\,, \tag{3.6}$$

where $\mathcal{V}_1(x)$ denotes the result of removing all covariant derivatives from the vertex $V_1(x)$. The covariant Taylor expansion introduces an additional factor of the parallel displacement propagator $I(x,x')$ (in the representation of the gauge group under which $\mathcal{V}_1$ transforms). This is denoted by the dashed line $I_4(x,x')$ in figure 2, while the solid lines $I_i(x,x'), i = 1,2,3$ denote the three parallel displacement propagators arising from the kernels. The filled square denotes the dependence on background fields, evaluated only at x'.

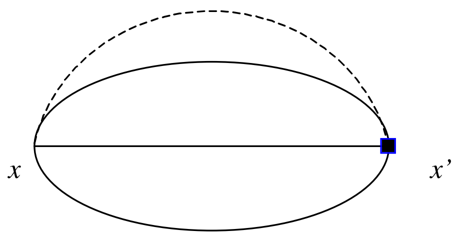

Figure 2: Accumulation of explicit background field dependence at one point.

The parallel displacement propagators $I_1(x,x'), \cdots, I_4(x,x')$ in figure 2 are in general in different representations of the gauge group. However, their product

$$I_4(x,x')\, I_1(x,x')\, I_2(x,x')\, I_3(x,x') \tag{3.7}$$

(gauge indices suppressed) is gauge invariant at x. Since each of these propagators satisfies the same equation (2.21), it is straightforward to show

$$I_4(x,x')\, I_1(x,x')\, I_2(x,x')\, I_3(x,x') = 1\,. \tag{3.8}$$

More precisely, the right-hand side of this relation is a product of Kronecker deltas. Thus the 'skeleton' of parallel displacement propagators collapses, leaving all background field dependence in the form of fields evaluated at x', depicted by the square in figure 2. There are also explicit factors of $(x - x')$ from the covariant Taylor expansion of fields in the vertex at x, and from covariant derivatives of the parallel displacement propagators. These can be traded for momentum derivatives by using

$$(x-x')^a\,\mathrm{e}^{\mathrm{i}k.(x-x')}F(k)=-\,\mathrm{i}\,F(k)\,\frac{\partial}{\partial k_a}\,\mathrm{e}^{\mathrm{i}k.(x-x')}=\mathrm{i}\,\mathrm{e}^{\mathrm{i}k.(x-x')}\,\frac{\partial}{\partial k_a}F(k)+\cdots\,, \tag{3.9}$$

where the dots denote a total derivative, which does not contribute to the momentum integral. This leaves all the dependence on $(x-x')$ in the form of the phase factor

$$\mathrm{e}^{\mathrm{i}\,(k_1+k_2+k_3).(x-x')}\,.$$

Since all background fields now depend only on x', the integral over x is

$$\int \mathrm{d}^d x\,\mathrm{e}^{\mathrm{i}(k_1+k_2+k_3).(x-x')}=(2\pi)^d\,\delta^d(k_1+k_2+k_3)\,, \tag{3.10}$$

and this delta function allows one of the three gaussian momentum integrals associated with the kernels to be eliminated. The procedure outlined above results in an expression for the contribution to the effective action from the 'fish' diagram in the form[7]

$$\begin{aligned}\int \mathrm{d}^d x' \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \int_0^\infty \mathrm{d}s_3 \int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d}\mathrm{e}^{-\mathrm{i}\,s_1 k_1^2}\mathrm{e}^{-\mathrm{i}\,s_2 k_2^2}\,\mathrm{e}^{-\mathrm{i}\,s_3 (k_1+k_2)^2}\,\times\\ \times F(s_1,s_2,s_3,k_1,k_2,\phi(x'))\,,\end{aligned} \tag{3.11}$$

where ϕ denotes background fields and their covariant derivatives. The momentum integrals here are just gaussian moments, which are straightforward to compute.

At three and higher loops, a similar procedure can be implemented, although the presence of extra vertices adds a complication. The aim is to transform the n-vertex diagram to a 'skeleton' which has the following properties:

(i) all explicit background field dependence is collected at a single vertex, say x_n;

(ii) the remaining vertices at $x_1,\ldots,x_{n-1}$ are connected to each other and to the vertex at x_n by (undifferentiated) parallel displacement propagators only;

(iii) the skeleton is gauge invariant at each vertex.

Consider, for example, the three-loop graph shown in figure 3.

Suppose we wish to move all the background field dependence to a single vertex, say at x_4. As a first step, we could choose to use the covariant Taylor expansion to express the background fields in the vertex $V_1(x_1)$ in terms of fields evaluated at x_2, resulting in the introduction of a parallel displacement propagator $I(x_1,x_2)$ in the representation of the gauge group under which the fields in V_1 transform. Using the steps outlined for the 'fish' diagram, it is also possible to manipulate the kernel $K(x_1,x_2|s_1)$ (or its derivatives if derivatives from the vertices act on it) and the explicit factors of (x_1-x_2) in the Taylor

[7] All covariant regularization schemes are consistent with our approach, therefore this issue does not require elaboration.

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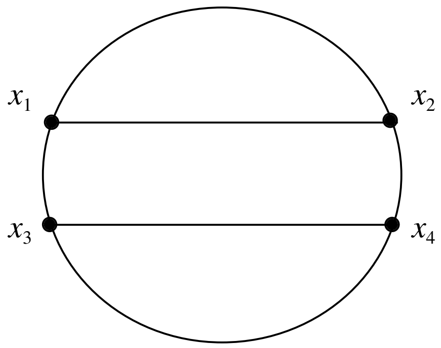

Figure 3: A three-loop graph.

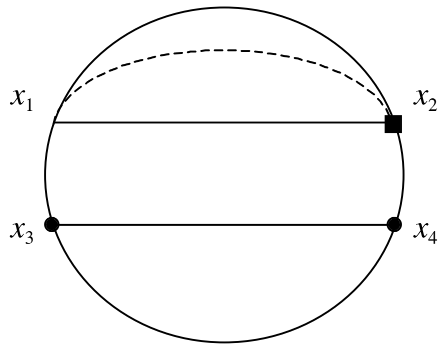

Figure 4: Removing explicit field dependence at x_1.

series so that all of the x_1 dependence is in the form $e^{\mathrm{i}k_1\cdot(x_1-x_2)}\, I(x_1, x_2)$. Here, the factor of $I(x_1, x_2)$ is in the representation under which $K(x_1, x_2|s_1)$ transforms. A new vertex, $\tilde{V}_2(x_2)$, has effectively been generated at x_2, consisting of the original background dependence of the vertex multiplied by the additional background dependence generated from $V_1(x_1)$ and $K(x_1, x_2|s_1)$. Diagrammatically, we have the situation in figure 4, where the circles denote original vertices and the square denotes the new effective vertex.

Repeating this procedure, the explicit dependence on background fields can be progessively moved through the diagram, at each step removing the field dependence at one vertex at the cost of introducing an additional parallel displacement propagator. It should be stressed that this procedure preserves manifest gauge invariance at each vertex. When all explicit background field dependence has been moved to a single point, the result is a skeleton as defined above.

Having produced the skeleton from the original Feynman diagram, it is now necessary to evaluate it. A very special case is that of a two-vertex skeleton, as illustrated in figure 5.

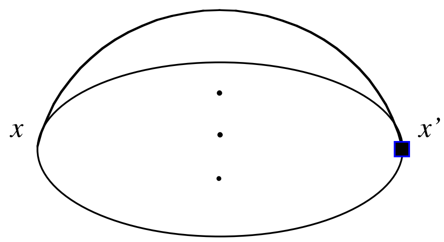

Figure 5: An arbitary two-vertex skeleton.

Here, the product of the parallel diplacement propagators is gauge invariant at x. Since all the propagators satisfy the same equation (2.21), it is straightforward to show that the product of parallel diplacement propagators simply collapses to a product of Kronecker deltas, leaving a gauge invariant contribution at a single point, x'. In the case where the skeleton contains three or more vertices, it is possible to systematically reduce it to a skeleton with only two vertices; the resulting diagram then collapses to a single point as above. We will explain the procedure for reducing a skeleton with n vertices to one with $n-1$ vertices; this process is then iterated to produce the two vertex skeleton.

The generic structure of the skeleton with n vertices is shown in figure 6, where x_1 is the point where all the explicit background field dependence is accumulated, and the lines are parallel displacement propagators.

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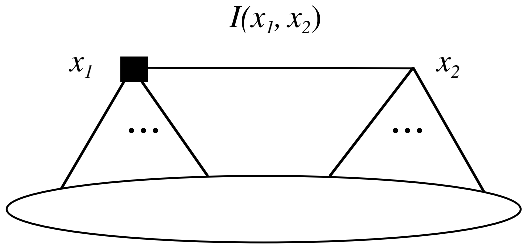

Figure 6: An arbitrary n-vertex skeleton, $n \geq 3$.

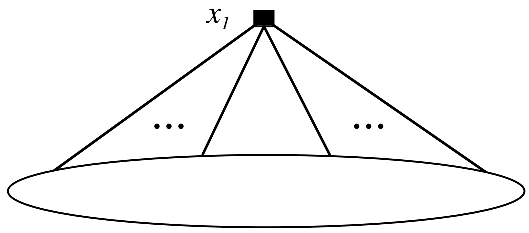

Figure 7: Reducing an n-vertex skeleton to an $(n-1)$-vertex skeleton, $n \geq 3$.

Focusing attention on a vertex adjacent to that at x_1, say that at x_2, the diagram has the structure

$$\mathrm{tr}\left(I(x_1, x_2)\,\chi(x_2, x_1)\right), \tag{3.12}$$

where $\chi(x_2, x_1)$ consists of the vertex at x_1 and products of parallel displacement propagators (dependence on points other than x_1 and x_2 has been suppressed). We can make use of the covariant Taylor series to expand $I(x_1, x_2)\,\chi(x_2, x_1)$ in the form

$$I(x_1, x_2)\,\chi(x_2, x_1) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(x_2 - x_1)^{a_1} \cdots (x_2 - x_1)^{a_n}\, \nabla_{a_1} \ldots \nabla_{a_n} \chi(y, x_1)\Big|_{y=x_1}. \tag{3.13}$$

By this means, all x_2 dependence of the skeleton is now in the form of factors $(x_2 - x_1)$. These factors can be replaced by momentum derivatives of the phase factor $\mathrm{e}^{\mathrm{i}k\cdot(x_1 - x_2)}$ associated with a Green's function in the original Feynman diagram, as in (3.9). This process reduces number of vertices in the skeleton by one, as illustrated in figure 7.

This completes our prescription for computing multi-loop contributions to the effective action in a manifestly covariant manner.

In the literature [24, 31], the configuration space approach has been the favoured means to achieve a covariant derivative expansion of the effective action. A possible explanation for this is a belief that the transition to momentum space should be incompatible with manifest gauge invariance, see e.g. [32]. As we have shown above, the latter is not the case. Our momentum space scheme is well-adapted to the derivative expansion of the effective action, in that it automatically links the factors of momentum to covariant derivatives of the background fields. Further, this approach is more economical in that there is no need to do separate configuration and momentum space integrals (configuration space integrals are just trivial). In contrast, the configuration space approach requires separate computation (by carrying out momentum integrals) of each Greeen's function in a diagram to the required order in the derivative expansion, and then the sewing together of the Green's functions with configuration space integrals. As noted earlier, the configuration space approach is well suited to the computation of the effective action in special backgrounds, as an exact expression for the Green's function can sometimes be computed. An example is a covariantly constant background; we treat this case in $\mathcal{N} = 1$ super Yang-Mills in section 6. It is worth noting that the momentum space scheme can easily be formulated in curved space due to the existence of a covariant Fourier integral [27].

4. Parallel displacement propagator in superspace

In four space-time dimensions, $d = 4$, we consider a flat global $\mathcal{N}$-extended superspace parametrized by variables $z^m = (x^m, \theta^\mu_i, \bar\theta^i_{\dot\mu})$, where $i = 1, \dots, \mathcal{N}$. We recall that the flat covariant derivatives $D_A = (\partial_a, D^i_\alpha, \bar D^{\dot\alpha}_i)$ are related to the supersymmetric Cartan 1-forms $\omega^A = (\omega^a, \omega^\alpha_i, \bar\omega^i_{\dot\alpha})$ by

$$\mathrm{d}z^M \, \partial_M = \omega^A \, D_A \,, \qquad \omega^A = \big(\mathrm{d}x^a - \mathrm{i}\, \mathrm{d}\theta_i \sigma^a \bar\theta^i + \mathrm{i}\, \theta_i \sigma^a \mathrm{d}\bar\theta^i, \mathrm{d}\theta^\alpha_i, \mathrm{d}\bar\theta^i_{\dot\alpha}\big) \,, \tag{4.1}$$

and satisfy the algebra

$$[D_A, D_B\} = T_{AB}{}^C \, D_C \,, \tag{4.2}$$

where the only non-vanishing components of the *constant* torsion $T_{AB}{}^C$ correspond to the anticommutator $\{D^i_\alpha, \bar D_{\dot\beta j}\} = -2\mathrm{i}\, \delta^i{}_j \, (\sigma^c)_{\alpha\dot\beta} \, \partial_c$.

To describe a Yang-Mills supermultiplet, we introduce gauge covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\alpha}_i)$,

$$\mathcal{D}_A = D_A + \mathrm{i}\, \Gamma_A(z) \,, \qquad [\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C \, \mathcal{D}_A + \mathrm{i}\, F_{AB}(z) \,, \tag{4.3}$$

where the connection Γ_A takes its values in the Lie algebra of a compact gauge group. The operators $\mathcal{D}_A$ possess the following gauge transformation law

$$\mathcal{D}_A \;\to\; \mathrm{e}^{\mathrm{i}\tau(z)} \, \mathcal{D}_A \, \mathrm{e}^{-\mathrm{i}\tau(z)} \,, \qquad \tau^\dagger = \tau \,, \tag{4.4}$$

with the gauge parameter $\tau(z)$ being arbitrary modulo the reality condition imposed.

In the case of $\mathcal{N} = 1$ supersymmetry, it is known that one can always bring, by applying a complex gauge transformation, the covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha})$ to the so-called covariantly chiral representation where

$$\mathcal{D}_\alpha = \mathrm{e}^{-V(z)} \, D_\alpha \, \mathrm{e}^{V(z)} \,, \qquad \bar{\mathcal{D}}_{\dot\alpha} = \bar D_{\dot\alpha} \,, \qquad V^\dagger = V \,, \tag{4.5}$$

with V the gauge prepotential. In the chiral representation (sometimes called the Λ-frame), the τ-transformations (4.4) are replaced by chiral Λ-transformations defined by

$$\mathcal{D}_A \;\to\; \mathrm{e}^{\mathrm{i}\Lambda(z)} \, \mathcal{D}_A \, \mathrm{e}^{-\mathrm{i}\Lambda(z)} \,, \qquad \bar D_{\dot\alpha} \Lambda = 0 \,. \tag{4.6}$$

In the present paper, we always work, for definiteness, in the τ-frame. It is also worth noting that one can define an analogue of the Λ-frame in the case of $\mathcal{N} = 2$ supersymmetry at the cost of embedding the standard $\mathcal{N} = 2$ superspace into the so-called harmonic superpace [55].

Let $z^M(t) = (z - z')^M \, t + z'^M$ be the straight line connecting two points z and z' in superspace, parametrized such that $z^M(0) = z'^M$ and $z^M(1) = z^M$. We then have $\dot z^M \, \partial_M = \zeta^A \, D_A$, where the two-point function $\zeta^A \equiv \zeta^A(z, z') = -\zeta^A(z', z)$ is

$$\zeta^A = \begin{cases} \zeta^a = (x - x')^a - \mathrm{i}(\theta - \theta')_i \sigma^a \bar\theta'^i + \mathrm{i}\theta'_i \sigma^a \left(\bar\theta - \bar\theta'\right)^i \,, \\ \zeta^\alpha_i = (\theta - \theta')^\alpha_i \,, \\ \bar\zeta^i_{\dot\alpha} = \left(\bar\theta - \bar\theta'\right)^i_{\dot\alpha} \,. \end{cases} \tag{4.7}$$

Given a gauge invariant superfield $\mathrm{U}(z)$, for $\mathrm{U}(t) = \mathrm{U}(z(t))$ we have

$$\frac{\mathrm{d}^n U}{\mathrm{d}t^n} = \zeta^{A_n} \dots \zeta^{A_1} \, D_{A_1} \dots D_{A_n} \, U \,. \tag{4.8}$$

This leads to a *supersymmetric Taylor series*

$$\mathrm{U}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \, \zeta^{A_n} \dots \zeta^{A_1} \, D'_{A_1} \dots D'_{A_n} \, \mathrm{U}(z') \,. \tag{4.9}$$

By definition, the parallel displacement propagator along the straight line, $I(z,z')$, is specified by the requirements:[8]

(i) the gauge transformation law

$$I(z,z') \to \mathrm{e}^{\mathrm{i}\tau(z)} \, I(z,z') \, \mathrm{e}^{-\mathrm{i}\tau(z')} \,; \tag{4.10}$$

(ii) the equation

$$\zeta^A \mathcal{D}_A \, I(z,z') = \zeta^A \Big(D_A + \mathrm{i} \, \Gamma_A(z) \Big) I(z,z') = 0 \,; \tag{4.11}$$

(iii) the boundary condition

$$I(z,z) = \mathbf{1} \,. \tag{4.12}$$

These imply the important relation

$$I(z,z') \, I(z',z) = \mathbf{1} \,. \tag{4.13}$$

We also have

$$\zeta^A \mathcal{D}'_A \, I(z,z') = \zeta^A \Big(D'_A \, I(z,z') - \mathrm{i} \, I(z,z') \, \Gamma_A(z') \Big) = 0 \,. \tag{4.14}$$

Further, using the identity

$$\zeta^B \, D_B \zeta^A = \zeta^A \,, \tag{4.15}$$

from (4.11) one deduces

$$\zeta^{A_n} \dots \zeta^{A_1} \, \mathcal{D}_{A_1} \dots \mathcal{D}_{A_n} \, I(z,z') = 0 \,. \tag{4.16}$$

The latter leads to

$$\mathcal{D}_{(A_1} \dots \mathcal{D}_{A_n\}} \, I(z,z') \Big|_{z=z'} = 0 \,, \qquad n \geq 1 \,, \tag{4.17}$$

where $(\dots\}$ means graded symmetrization of n indices (with a factor of $1/n!$).

Let $\Psi(z)$ be a superfield transforming in some representation of the gauge group,

$$\Psi(z) \to \mathrm{e}^{\mathrm{i}\tau(z)} \, \Psi(z) \,. \tag{4.18}$$

[8] The unique solution to the requirements (i)–(iii) is given by $I(z,z') = \mathrm{P} \, \exp\,[-\mathrm{i} \int_{z'}^{z} \mathrm{d}t \, \zeta^A \Gamma_A(z(t))]$, where the integration is carried out along the straight line connecting the points z' and z. For an arbitrary path $\gamma : t \to z^M(t)$ in superspace, such supersymmetric phase factors of the form $\Sigma[\gamma] = \mathrm{P} \, \exp\,[-\mathrm{i} \int_\gamma \omega \cdot \Gamma(z(t))]$, were considered in [52, 53, 54].

Then $\mathrm{U}(z) \equiv I(z', z)\, \Psi(z)$ is gauge invariant with respect to z, and therefore we are in a position to apply the Taylor expansion (4.9):

$$\Psi(z) = I(z, z') \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{A_n} \ldots \zeta^{A_1} \mathcal{D}_{A_1} \ldots \mathcal{D}_{A_n} \left(I(z', w)\, \Psi(w) \right) \Big|_{w=z'} . \tag{4.19}$$

Because of the properties of $I(z, z')$, this is equivalent to the covariant Taylor series

$$\Psi(z) = I(z, z') \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{A_n} \ldots \zeta^{A_1} \mathcal{D}'_{A_1} \ldots \mathcal{D}'_{A_n} \Psi(z') . \tag{4.20}$$

The covariant Taylor expansion can be applied to $\mathcal{D}_B I(z, z')$ considered as a superfield at z,

$$\mathcal{D}_B I(z, z') = I(z, z') \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{A_n} \ldots \zeta^{A_1} \mathcal{D}_{A_1} \ldots \mathcal{D}_{A_n} \mathcal{D}_B\, I(w, z') \Big|_{w=z'} . \tag{4.21}$$

As is shown in the appendix A, this is equivalent to

$$\begin{aligned} \mathcal{D}_B I(z, z') = {} & \mathrm{i}\, I(z, z') \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \times \\ & \Big\{ n\, \zeta^{A_n} \ldots \zeta^{A_1} \mathcal{D}'_{A_1} \ldots \mathcal{D}'_{A_{n-1}} F_{A_n\, B}(z') + \\ & + \frac{1}{2} (n-1)\, \zeta^{A_n} T_{A_n\, B}{}^{C}\, \zeta^{A_{n-1}} \ldots \zeta^{A_1} \mathcal{D}'_{A_1} \ldots \mathcal{D}'_{A_{n-2}} F_{A_{n-1}\, C}(z') \Big\} . \end{aligned} \tag{4.22}$$

Another form for this result is

$$\begin{aligned} \mathcal{D}_B I(z, z') = {} & \mathrm{i} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \times \\ & \times \Big\{ -\zeta^{A_n} \ldots \zeta^{A_1} \mathcal{D}_{A_1} \ldots \mathcal{D}_{A_{n-1}} F_{A_n\, B}(z) \\ & + \frac{1}{2} (n-1)\, \zeta^{A_n} T_{A_n\, B}{}^{C}\, \zeta^{A_{n-1}} \ldots \zeta^{A_1} \mathcal{D}_{A_1} \ldots \mathcal{D}_{A_{n-2}} F_{A_{n-1}\, C}(z) \Big\}\, I(z, z') . \end{aligned} \tag{4.23}$$

For the point $z^M(t)$ on the straight line $z^M(t) = (z - z')^M\, t + z'^M$, we introduce the notation

$$\zeta^A(t) \equiv \zeta^A(z(t), z') = t\, \zeta^A , \tag{4.24}$$

with the two-point function $\zeta^A(z, z')$ as defined earlier. Then the results (4.22) and (4.23) can be expressed in integral form as

$$\begin{aligned} \mathcal{D}_B\, I(z, z') = {} & \mathrm{i} \int_0^1 \mathrm{d}t\, t\, I(z, z(t))\, \zeta^A\, F_{AB}(z(t))\, I(z(t), z') + \\ & + \frac{\mathrm{i}}{2} \int_0^1 \mathrm{d}t_1\, t_1\, I(z, z(t_1))\, \zeta^A\, T_{AB}{}^C \int_0^1 \mathrm{d}t_2\, t_2\, I(z(t_1), z(t_2 t_1)) \times \\ & \times t_1\, \zeta^D\, F_{DC}(z(t_2 t_1))\, I(z(t_2 t_1), z') . \end{aligned} \tag{4.25}$$

The latter relation seems to be equivalent to the one derived in [54].

By analogy with the non-supersymmetric case, let us fix some point z' in superspace and consider the gauge transformation generated by

$$\mathrm{e}^{\mathrm{i}\tau(z)} = I(z', z)\,, \qquad \mathrm{e}^{\mathrm{i}\tau(z')} = \mathbf{1}\,. \tag{4.26}$$

Applying this gauge transformation to $I(z, z')$, in accordance with (4.10), we end up with the superspace Fock-Schwinger gauge [57]

$$I(z, z') = \mathbf{1} \quad \Longleftrightarrow \quad \zeta^A\,\Gamma_A(z) = 0\,. \tag{4.27}$$

With this gauge choice, eq. (4.22) tells us that the superconnection is

$$\begin{aligned}
\Gamma_B(z) = & \sum_{n=1}^{\infty} \frac{1}{(n+1)!}\times \\
& \times\Big\{n\,\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}'_{A_1}\ldots\mathcal{D}'_{A_{n-1}}F_{A_n\,B}(z') + \\
& \quad + \frac{1}{2}(n-1)\,\zeta^{A_n}T_{A_n\,B}{}^{C}\,\zeta^{A_{n-1}}\ldots\zeta^{A_1}\mathcal{D}'_{A_1}\ldots\mathcal{D}'_{A_{n-2}}F_{A_{n-1}\,C}(z')\Big\}\,.
\end{aligned} \tag{4.28}$$

In the case of $\mathcal{N} = 1$ supersymmetry,[9] the gauge covariant derivatives satisfy the following (anti)commutation relations:

$$\begin{aligned}
\{\mathcal{D}_a, \mathcal{D}_\beta\} &= \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0\,, \qquad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -2\mathrm{i}\,\mathcal{D}_{\alpha\dot\beta}\,, \\
[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}] &= 2\mathrm{i}\varepsilon_{\alpha\beta}\,\bar{W}_{\dot\beta}\,, \qquad [\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\,W_\beta\,, \\
[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] &= \mathrm{i}F_{\alpha\dot\alpha,\beta\dot\beta} = -\varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}_{\dot\alpha}\bar{W}_{\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\,\mathcal{D}_\alpha W_\beta\,,
\end{aligned} \tag{4.29}$$

with the spinor field strengths W_α and $\bar{W}_{\dot\alpha}$ obeying the Bianchi identities

$$\bar{\mathcal{D}}_{\dot\alpha} W_\alpha = 0\,, \qquad \mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}\,. \tag{4.30}$$

For a covariantly constant gauge field,

$$\mathcal{D}_a\,W_\beta = 0\,, \tag{4.31}$$

the relation (4.22) is equivalent to the following:

$$\begin{aligned}
\mathcal{D}_{\beta\dot\beta}I(z, z') = & \; I(z, z')\Big(-\frac{\mathrm{i}}{4}\zeta^{\dot\alpha\alpha}F_{\alpha\dot\alpha,\beta\dot\beta}(z') - \mathrm{i}\,\zeta_\beta\bar{W}_{\dot\beta}(z') + \mathrm{i}\,\bar\zeta_{\dot\beta}W_\beta(z') + \\
& \qquad + \frac{2\mathrm{i}}{3}\,\bar\zeta_{\dot\beta}\zeta^\alpha\mathcal{D}_\alpha W_\beta(z') + \frac{2\mathrm{i}}{3}\,\zeta_\beta\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{W}_{\dot\beta}(z')\Big) \\
= & \Big(-\frac{\mathrm{i}}{4}\zeta^{\dot\alpha\alpha}F_{\alpha\dot\alpha,\beta\dot\beta}(z) - \mathrm{i}\,\zeta_\beta\bar{W}_{\dot\beta}(z) + \mathrm{i}\,\bar\zeta_{\dot\beta}W_\beta(z) - \\
& \qquad - \frac{\mathrm{i}}{3}\,\bar\zeta_{\dot\beta}\zeta^\alpha\mathcal{D}_\alpha W_\beta(z) - \frac{\mathrm{i}}{3}\,\zeta_\beta\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{W}_{\dot\beta}(z)\Big)\, I(z, z')\,;
\end{aligned} \tag{4.32}$$

[9] Our $\mathcal{N} = 1$ notation and conventions correspond to [44].

$$
\begin{aligned}
\mathcal{D}_\beta I(z,z') &= I(z,z')\bigg(\frac{1}{12}\,\bar\zeta^{\dot\beta}\zeta^{\alpha\dot\alpha}F_{\alpha\dot\alpha,\beta\dot\beta}(z') - \mathrm{i}\,\zeta_{\beta\dot\beta}\Big\{\frac{1}{2}\bar W^{\dot\beta}(z') - \frac{1}{3}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\beta}(z')\Big\} + \\
&\qquad + \frac{1}{3}\,\zeta_\beta\bar\zeta_{\dot\beta}\bar W^{\dot\beta}(z') + \frac{1}{3}\bar\zeta^2 \times \\
&\qquad \times \Big\{W_\beta(z') + \frac{1}{2}\zeta^\alpha\mathcal{D}_\alpha W_\beta(z') - \frac{1}{4}\zeta_\beta\mathcal{D}^\alpha W_\alpha(z')\Big\}\bigg) \\
&= \Big(\frac{1}{12}\,\bar\zeta^{\dot\beta}\zeta^{\alpha\dot\alpha}\,F_{\alpha\dot\alpha,\beta\dot\beta}(z) - \frac{\mathrm{i}}{2}\,\zeta_{\beta\dot\beta}\Big\{\bar W^{\dot\beta}(z) + \frac{1}{3}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\beta}(z)\Big\} + \frac{1}{3}\,\zeta_\beta\bar\zeta_{\dot\beta}\bar W^{\dot\beta}(z) \\
&\qquad + \frac{1}{3}\bar\zeta^2\Big\{W_\beta(z) - \frac{1}{2}\zeta^\alpha\mathcal{D}_\alpha W_\beta(z) + \frac{1}{4}\zeta_\beta\mathcal{D}^\alpha W_\alpha(z)\Big\}\Big)\, I(z,z')\ ; \qquad (4.33)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{D}}_{\dot\beta} I(z,z') &= I(z,z')\bigg(-\frac{1}{12}\,\zeta^{\beta}\zeta^{\alpha\dot\alpha}\,F_{\alpha\dot\alpha,\beta\dot\beta}(z') - \mathrm{i}\,\zeta_{\beta\dot\beta}\,\Big\{\frac{1}{2}W^{\beta}(z') + \frac{1}{3}\zeta^{\alpha}\mathcal{D}_{\alpha}W^{\beta}(z')\Big\} - \\
&\qquad - \frac{1}{3}\,\bar\zeta_{\dot\beta}\zeta^\beta W_\beta(z') - \\
&\qquad - \frac{1}{3}\zeta^2\,\Big\{\bar W_{\dot\beta}(z') - \frac{1}{2}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar W_{\dot\beta}(z') + \frac{1}{4}\bar\zeta_{\dot\beta}\bar{\mathcal{D}}^{\dot\alpha}\bar W_{\dot\alpha}(z')\Big\}\bigg) \\
&= \bigg(-\frac{1}{12}\,\zeta^\beta\zeta^{\alpha\dot\alpha}F_{\alpha\dot\alpha,\beta\dot\beta}(z) - \frac{\mathrm{i}}{2}\,\zeta_{\beta\dot\beta}\Big\{W^\beta(z) - \frac{1}{3}\zeta^\alpha\mathcal{D}_\alpha W^\beta(z)\Big\} - \frac{1}{3}\,\bar\zeta_{\dot\beta}\zeta^\beta W_\beta(z) - \\
&\qquad - \frac{1}{3}\zeta^2\,\Big\{\bar W_{\dot\beta}(z) + \frac{1}{2}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar W_{\dot\beta}(z) - \frac{1}{4}\bar\zeta_{\dot\beta}\bar{\mathcal{D}}^{\dot\alpha}\bar W_{\dot\alpha}(z)\Big\}\bigg) I(z,z')\,. \qquad (4.34)
\end{aligned}
$$

5. Green's functions in superspace

In the remaining sections of this paper, we will concentrate on $\mathcal{N} = 1$ supersymmetric Yang-Mills theories. A typical Green's function $G(z,z')$ will be that of an unconstrained superfield transforming in some (real) representation of the gauge group. The Green's function satisfies the equation

$$
\Box_z\, G(z,z') = -\delta^8(z-z')\,\mathbf{1}\,, \qquad \delta^8(z-z') = \delta^4(x-x')\,(\theta-\theta')^2\,\big(\bar\theta-\bar\theta'\big)^2\,, \tag{5.1}
$$

and the Feynman boundary conditions. Here the covariant d'Alembertian $\Box$ is [15]

$$
\begin{aligned}
\Box &= \mathcal{D}^a\mathcal{D}_a - W^\alpha\mathcal{D}_\alpha + \bar W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} \qquad (5.2)\\
&= -\frac{1}{8}\mathcal{D}^\alpha\bar{\mathcal{D}}^2\mathcal{D}_\alpha + \frac{1}{16}\,\{\mathcal{D}^2,\bar{\mathcal{D}}^2\} - W^\alpha\mathcal{D}_\alpha - \frac{1}{2}(\mathcal{D}^\alpha W_\alpha) \\
&= -\frac{1}{8}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}^2\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{16}\,\{\mathcal{D}^2,\bar{\mathcal{D}}^2\} + \bar W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2}\,\big(\bar{\mathcal{D}}_{\dot\alpha}\bar W^{\dot\alpha}\big)\,.
\end{aligned}
$$

Our subsequent considerations require only a minor modification if the operator $\Box_z$ in (5.1) is replaced with $\Box_z - \mathcal{P}(z)$, with $\mathcal{P}$ a local matrix function of the gauge field. For simplicity, we set $\mathcal{P} = 0$.

In complete analogy with the non-supersymmetric case, we introduce the proper-time representation of $G(z,z')$. The corresponding heat kernel,

$$
K(z,z'|s) = \mathrm{e}^{\mathrm{i}s\,(\Box+\mathrm{i}\,\varepsilon)}\,\delta^8(z-z')\,\mathbf{1}\,, \qquad \varepsilon\to+0\,, \tag{5.3}
$$

will be the main object of interest. Its gauge transformation law is

$$
K(z,z'|s)\ \to\ \mathrm{e}^{\mathrm{i}\tau(z)}\,K(z,z'|s)\,\mathrm{e}^{-\mathrm{i}\tau(z')}\,. \tag{5.4}
$$

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As in the non-supersymmetric case, it is useful to make use of the Fourier representation

$$\delta^8(z-z') = \frac{1}{\pi^4}\int \mathrm{d}^4k \int \mathrm{d}^2\kappa \int \mathrm{d}^2\bar\kappa\; \mathrm{e}^{\mathrm{i}\,[k^a\zeta_a + \kappa^\alpha\zeta_\alpha + \bar\kappa_{\dot\alpha}\bar\zeta^{\dot\alpha}]}\,, \tag{5.5}$$

where the supersymmetric interval ζ^A is defined in eq. (4.7), and the integration variables k^a and $(\kappa^\alpha, \bar\kappa_{\dot\alpha})$ are c-numbers and a-numbers, respectively. In order to keep the gauge transformation law (5.4) manifest, we should actually represent the full delta-function in the form

$$\delta^8(z-z')\,\mathbf{1} = \frac{1}{\pi^4}\int \mathrm{d}^4k \int \mathrm{d}^2\kappa \int \mathrm{d}^2\bar\kappa\; \mathrm{e}^{\mathrm{i}\,[k^a\zeta_a + \kappa^\alpha\zeta_\alpha + \bar\kappa_{\dot\alpha}\bar\zeta^{\dot\alpha}]}\, I(z,z')\,, \tag{5.6}$$

with $I(z,z')$ the parallel dispacement propagator. As a result, the heat kernel takes the form

$$\begin{aligned} K(z,z'|s) &= \hat K(z,z'|s)\, I(z,z')\,, \\ \hat K(z,z'|s) &\equiv \frac{1}{\pi^4}\int \mathrm{d}^4k \int \mathrm{d}^2\kappa \int \mathrm{d}^2\bar\kappa\; \mathrm{e}^{\mathrm{i}\,[k^a\zeta_a + \kappa^\alpha\zeta_\alpha + \bar\kappa_{\dot\alpha}\bar\zeta^{\dot\alpha}]}\, \mathrm{e}^{\mathrm{i}s\,[X^aX_a - W^\alpha X_\alpha + \bar W_{\dot\alpha}\bar X^{\dot\alpha}]}\,, \end{aligned} \tag{5.7}$$

where

$$X_a = \mathcal{D}_a + \mathrm{i}k_a\,, \qquad X_\alpha = \mathcal{D}_\alpha + \mathrm{i}\kappa_\alpha - k^a(\sigma_a)_{\alpha\dot\alpha}\bar\zeta^{\dot\alpha}\,, \qquad \bar X^{\dot\alpha} = \bar{\mathcal{D}}^{\dot\alpha} + \mathrm{i}\bar\kappa^{\dot\alpha} - k^a(\tilde\sigma_a)^{\dot\alpha\alpha}\zeta_\alpha\,. \tag{5.8}$$

With respect to the gauge group, the operator $\hat K(z,z'|s)$ transforms as

$$\hat K(z,z'|s) \to \mathrm{e}^{\mathrm{i}\tau(z)}\,\hat K(z,z'|s)\,\mathrm{e}^{-\mathrm{i}\tau(z)}\,. \tag{5.9}$$

For any gauge invariant superfield $\Omega(z)$ of compact support in space-time, we can establish the following operator identity

$$\hat K\left(z,z'|s\right)\cdot\hat\Omega(z) = \hat\Omega(z')\cdot\hat K\left(z,z'|s\right)\,, \qquad \hat\Omega(z) \equiv \Omega(z)\,\mathbf{1}\,. \tag{5.10}$$

To prove this, one should first introduce a generalized Fourier representation of Ω,

$$\Omega(z) = \int \mathrm{d}^4p \int \mathrm{d}^2\rho \int \mathrm{d}^2\bar\rho\; \mathrm{e}^{\mathrm{i}\,[p^ax_a + \rho^\alpha\theta_\alpha + \bar\rho_{\dot\alpha}\bar\theta^{\dot\alpha}]}\,\Omega(p,\rho,\bar\rho)\,, \tag{5.11}$$

then push the "plane wave" $\exp(\mathrm{i}\,[p^ax_a + \rho^\alpha\theta_\alpha + \bar\rho_{\dot\alpha}\bar\theta^{\dot\alpha}])$ through the operatorial exponential $\exp(\mathrm{i}s\,[X^aX_a - W^\alpha X_\alpha + \bar W_{\dot\alpha}\bar X^{\dot\alpha}])$ in the integral representation (5.7) of $\hat K(z,z'|s)$, and finally introduce new integration variables by the rule

$$k'^a = k^a + p^a\,, \qquad \kappa'^\alpha = \kappa^\alpha + \rho^\alpha - \mathrm{i}\,p^a\left(\bar\theta'\tilde\sigma_a\right)^\alpha\,, \qquad \bar\kappa'_{\dot\alpha} = \bar\kappa_{\dot\alpha} + \bar\rho_{\dot\alpha} - \mathrm{i}\,p^a\left(\theta'\sigma_a\right)_{\dot\alpha}\,. \tag{5.12}$$

Relation (5.10) implies

$$\hat K(z,z'|s) = \,:\!\mathrm{e}^{-\zeta^A\mathcal{D}_A}\!:\,\hat B(z,z'|s)\,, \tag{5.13}$$

where the two-point matrix $\hat B(z,z'|s)$ is a functional of the gauge superfield such that

$$\hat B(z,z'|s)\cdot\hat\Omega(z) = \hat\Omega(z)\cdot\hat B(z,z'|s). \tag{5.14}$$

Therefore $\hat{B}(z,z'|s)$ is not a differential operator. By definition, the operator $:\mathrm{e}^{-\zeta^A \mathcal{D}_A}:$ acts on a gauge covariant superfield $\Psi(z)$ (transforming in some representation of the gauge group) as follows

$$:\mathrm{e}^{-\zeta^A \mathcal{D}_A}:\Psi(z) = I(z,z')\,\Psi(z')\,. \tag{5.15}$$

If $\Psi(z)$ admits a covariant Taylor expansion, then

$$:\mathrm{e}^{-\zeta^A \mathcal{D}_A}:\Psi(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(-1)^n\,\zeta^{A_n}\ldots\zeta^{A_1}\,\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_n}\,\Psi(z)\,. \tag{5.16}$$

Evaluation of the heat kernel (5.7) can be carried out in a manner almost identical to that outlined at the end of section 2 for the non-supersymmetric case. The result is

$$K(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\mathrm{e}^{\mathrm{i}\zeta^a\zeta_a/4s}\,F(z,z'|s)\,, \qquad F(z,z'|s) = \sum_{n=0}^{\infty} a_n(z,z')\,(\mathrm{i}s)^n\,, \tag{5.17}$$

where

$$\begin{aligned} a_0(z,z') &= :\mathrm{e}^{-\zeta^a \mathcal{D}_a}:\delta^4(\theta-\theta')\,I(z,z') = \delta^4(\theta-\theta'):\mathrm{e}^{-\zeta^a \mathcal{D}_a}:I(z,z') \\ &= \delta^4(\theta-\theta'):\mathrm{e}^{-\zeta^A \mathcal{D}_A}:I(z,z') = \delta^4(\theta-\theta')\,I(z,z')\,. \end{aligned} \tag{5.18}$$

Here $\delta^4(\theta-\theta') = (\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2$, and the operator $:\mathrm{e}^{-\zeta^a \mathcal{D}_a}:$ is defined similarly to (2.15).

Along with the propagator $G(z,z')$ so far analysed, covariant supergraphs in $\mathcal{N}=1$ super Yang-Mills theories also involve (anti)chiral Green's functions[10] $G_\mp(z,z')$ associated with the covariantly chiral d'Alembertian

$$\Box_+ = \mathcal{D}^a\mathcal{D}_a - W^\alpha\mathcal{D}_\alpha - \frac{1}{2}\,(\mathcal{D}^\alpha W_\alpha)\,, \qquad \Box_+\Phi = \frac{1}{16}\,\bar{\mathcal{D}}^2\mathcal{D}^2\Phi\,, \qquad \bar{\mathcal{D}}_{\dot\alpha}\Phi = 0\,, \tag{5.19}$$

and the covariantly antichiral d'Alembertian

$$\Box_- = \mathcal{D}^a\mathcal{D}_a + \bar{W}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2}\left(\bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}\right), \qquad \Box_-\bar\Phi = \frac{1}{16}\,\mathcal{D}^2\bar{\mathcal{D}}^2\bar\Phi\,, \qquad \mathcal{D}_\alpha\bar\Phi = 0\,. \tag{5.20}$$

Associated with the chiral Green's function $G_+(z,z')$ is the covariantly chiral heat kernel

$$\begin{aligned} K_+(z,z'|s) &= \mathrm{e}^{\mathrm{i}s\,(\Box_+ + \mathrm{i}\,\varepsilon)}\left(-\frac{1}{4}\bar{\mathcal{D}}^2\right)\delta^8(z-z')\,\mathbf{1}\,, \qquad \varepsilon \to +0\,, \\ \bar{\mathcal{D}}_{\dot\alpha}K_+(z,z'|s) &= \bar{\mathcal{D}}'_{\dot\alpha}K_+(z,z'|s) = 0\,, \end{aligned} \tag{5.21}$$

and similarly in the antichiral case.

In the case of an on-shell background gauge superfield,

$$\mathcal{D}^\alpha W_\alpha = 0\,, \tag{5.22}$$

which is often sufficient for applications, the (anti)chiral kernels $K_\mp(z,z'|s)$ do not require a separate treatment. Indeed, we have

$$\Box\,\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}^2\,\Box\,, \qquad \Box\,\bar{\mathcal{D}}^2 = \Box_+\,\bar{\mathcal{D}}^2\,, \tag{5.23}$$

and therefore

$$K_+(z,z'|s) = -\frac{1}{4}\bar{\mathcal{D}}^2\,K(z,z'|s)\,. \tag{5.24}$$

The strategy explained in section 3 for computing multi-loop graphs is readily extended to the case of supergraphs by making use of the results of this section.

[10] The proper-time representation for these Green's functions was developed in [58, 42], see also [59, 60, 44].

6. Superpropagator in a covariantly constant field

In this section, we evaluate the heat kernel (5.3) in the the case of a covariantly constant gauge superfield satisfying eq. (4.31). Taken together with the Bianchi identities, this condition implies that the Yang-Mills supermultiplet belongs to the Cartan subalgebra of the gauge group. For this background gauge superfield, the heat kernel (5.3) was originally computed by Ohrndorf [56] using the Fock-Schwinger gauge in superspace [57]. Here we derive, for the first time, this exact solution in a manifestly gauge covariant way.

It follows from (4.29) that

$$\mathcal{D}_a W_\beta = 0 \quad \Longrightarrow \quad \left[\mathcal{D}_a, W^\beta \mathcal{D}_\beta - \bar{W}_{\dot\beta} \bar{\mathcal{D}}^{\dot\beta}\right] = 0\,. \tag{6.1}$$

This identity allows a convenient factorization of the kernel in the form

$$K(z,z'|s) = \mathrm{U}(s)\,\mathrm{e}^{\mathrm{i}s\,\mathcal{D}^a\mathcal{D}_a}\,\delta^8(z-z')\,\mathbf{1}\,, \qquad \mathrm{U}(s) = \mathrm{e}^{-\mathrm{i}s(W^\alpha\mathcal{D}_\alpha - \bar{W}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha})}\,. \tag{6.2}$$

As a result, it proves efficient to use a Fourier transformation of only the bosonic part of the superspace delta function,

$$\delta^8(z-z') = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}k^a\zeta_a}\,\zeta^2\bar{\zeta}^2\,, \tag{6.3}$$

where ζ^A is defined in eq. (4.7).

The kernel of interest (5.3) is then obtained by the action of the operator $\mathrm{U}(s)$ on the simpler kernel

$$\tilde{K}(z,z'|s) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}k^a\zeta_a}\,\mathrm{e}^{\mathrm{i}s(\mathcal{D}+\mathrm{i}k)^2}\,\zeta^2\,\bar{\zeta}^2\,I(z,z')\,, \tag{6.4}$$

where $I(z,z')$ is the parallel displacement operator. The operator $\mathrm{U}(s)$ acts to "shift" the ζ^A dependence of $\tilde{K}(z,z'|s)$. With the notation $N_\alpha{}^\beta = \mathcal{D}_\alpha W^\beta$,

$$\begin{aligned}
\mathrm{U}(s)\,W^\alpha\;\mathrm{U}(-s) &\equiv W^\alpha(s) = W^\beta\left(\mathrm{e}^{-\mathrm{i}sN}\right)_\beta{}^\alpha\,,\\
\mathrm{U}(s)\,\zeta^\alpha\;\mathrm{U}(-s) &\equiv \zeta^\alpha(s) = \zeta^\alpha + W^\beta\left(\left(\mathrm{e}^{-\mathrm{i}sN}-1\right)N^{-1}\right)_\beta{}^\alpha\,,\\
\mathrm{U}(s)\,\zeta_{\alpha\dot\alpha}\;\mathrm{U}(-s) &\equiv \zeta_{\alpha\dot\alpha}(s) = \zeta_{\alpha\dot\alpha} - 2\int_0^s \mathrm{d}t\,\left(W_\alpha(t)\bar{\zeta}_{\dot\alpha}(t) + \zeta_\alpha(t)\bar{W}_{\dot\alpha}(t)\right)\,.
\end{aligned} \tag{6.5}$$

The kernel $\tilde{K}(z,z'|s)$ satisifies the equation

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s} + \mathcal{D}^a\mathcal{D}_a\right)\tilde{K}(z,z'|s) = 0 \tag{6.6}$$

with the boundary condition $\lim_{s\to 0}\tilde{K}(z,z'|s) = \delta^8(z-z')\,\mathbf{1}$. The identity

$$\begin{aligned}
0 &= \int \mathrm{d}^4k\,\frac{\partial}{\partial k^a}\left(\mathrm{e}^{\mathrm{i}k^b\zeta_b}\,\mathrm{e}^{\mathrm{i}s(\mathcal{D}+\mathrm{i}k)^2}\,\zeta^2\,\bar{\zeta}^2\,I(z,z')\right) \qquad (6.7)\\
&= \mathrm{i}\zeta_a\tilde{K}(z,z'|s) - 2s\int \mathrm{d}^4k\mathrm{e}^{\mathrm{i}k^b\zeta_b}\int_0^1 \mathrm{d}t\mathrm{e}^{\mathrm{i}st(\mathcal{D}+\mathrm{i}k)^2}(\mathcal{D}_a+\mathrm{ik_a})\mathrm{e}^{-\mathrm{ist}(\mathcal{D}+\mathrm{ik})^2}\mathrm{e}^{\mathrm{is}(\mathcal{D}+\mathrm{ik})^2}\zeta^2\bar{\zeta}^2\mathrm{I}(\mathrm{z},\mathrm{z}')
\end{aligned}$$

can be used to show that

$$\mathcal{D}_a\,\tilde{K}(z,z'|s) = \mathrm{i}\left(\frac{F}{\mathrm{e}^{2sF}-1}\right)_{ab}\zeta^b\,\tilde{K}(z,z'|s)\,. \tag{6.8}$$

Differentiating (6.8) again allows the right hand side of (6.6) to be expressed in terms of $\tilde{K}$, and equation (6.6) can then be integrated in the form

$$\tilde{K}(z,z'|s) = -\frac{\mathrm{i}}{16\pi^2}\,\det\left(\frac{2\,F}{\mathrm{e}^{2sF}-1}\right)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{4}\zeta^a(F\coth(sF))_{ab}\zeta^b}\,\zeta^2\,\bar{\zeta}^2\,C(z,z')\,, \tag{6.9}$$

where the determinant is computed with respect to the Lorentz indices. Here, $C(z,z')$ is an integration constant which must transform appropriately under the gauge group and satisfy the boundary condition $C(z,z) = \mathbf{1}$. Substituting (6.9) into (6.8) yields the further condition

$$\zeta^2\,\bar{\zeta}^2\,\mathcal{D}_a C(z,z') = -\frac{\mathrm{i}}{2}\,\zeta^2\,\bar{\zeta}^2\,F_{ab}\zeta^b\,C(z,z')\,, \tag{6.10}$$

as the action of $\mathcal{D}_a$ on the expression $\exp[\frac{\mathrm{i}}{4}\zeta^a(F\coth(sF))_{ab}\zeta^b]$ produces the symmetric part of $\left(\frac{\mathrm{i}\,F}{\mathrm{e}^{2sF}-1}\right)$ on the right hand side of (6.8), but not the antisymmetric part $(-\frac{\mathrm{i}}{2}F)$. In accordance with (4.32), we can choose $C(z,z') = I(z,z')$. As a result we arrive at the kernel

$$K(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\det\left(\frac{2\,s\,F}{\mathrm{e}^{2sF}-1}\right)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{4}\zeta^a(s)(F\coth(sF))_{ab}\zeta^b(s)}\times\zeta^2(s)\,\bar{\zeta}^2(s)\;\mathrm{U}(s)\,I(z,z')\,, \tag{6.11}$$

with $\zeta^A(s)$ defined in (6.5). The final ingredient is

$$\mathrm{U}(s)\,I(z,z') = \exp\left\{\int_0^s \mathrm{d}t\,\Xi(\zeta(t),W(t),\bar{W}(t))\right\}\,I(z,z')\,, \tag{6.12}$$

where

$$\begin{aligned}
\Xi\left(\zeta(s),W(s),\bar{W}(s)\right) &= \mathrm{U}(s)\,\Xi\left(\zeta,W,\bar{W}\right)\mathrm{U}(-s)\,,\\
\Xi\left(\zeta,W,\bar{W}\right) &= \frac{1}{12}\,\zeta^{\dot{\alpha}\alpha}\left(W^\beta\bar{\zeta}^{\dot{\beta}} - \zeta^\beta\bar{W}^{\dot{\beta}}\right)\left(\varepsilon_{\beta\alpha}\,\bar{\mathcal{D}}_{\dot{\beta}}\bar{W}_{\dot{\alpha}} - \varepsilon_{\dot{\beta}\dot{\alpha}}\,\mathcal{D}_\beta W_\alpha\right) -\\
&\quad -\frac{2\mathrm{i}}{3}\,\zeta W\,\bar{\zeta}\bar{W} - \frac{\mathrm{i}}{3}\,\zeta^2\left(\bar{W}^2 - \frac{1}{4}\bar{\zeta}\bar{\mathcal{D}}\,\bar{W}^2 + \frac{1}{4}\bar{\zeta}\bar{W}\,\bar{\mathcal{D}}\bar{W}\right) -\\
&\quad -\frac{\mathrm{i}}{3}\,\bar{\zeta}^2\left(W^2 - \frac{1}{4}\zeta\mathcal{D}\,W^2 + \frac{1}{4}\zeta W\,\mathcal{D}W\right)\,.
\end{aligned} \tag{6.13}$$

If the background gauge superfield is further constrained to satisfy the equation of motion (5.22), then from (6.11) we can immediately derive the chiral kernel $K_+(z,z'|s)$ using the prescription (5.24). The result for $K_+(z,z'|s)$ can be shown to agree with the exact superpropagator computed in [60] (see also a related work [59]), although the final relation in (the English translation of) [60] contains a misprint.

In conclusion, we would like to comment on the relationship between $K_+(z,z'|s)$ and the chiral propagator used in [61] for a calculation in the context of the matrix model/gauge theory correspondence [62]. The authors of [61] evaluate chiral loop diagrams in an on-shell covariantly constant SYM background, as specified by eqs. (4.31) and (5.22), which is then further simplified by formally setting $\bar W_{\dot\alpha} = 0$ and $F_{ab} = 0$ while keeping $W_\alpha \neq 0$. For such a background, in the chiral representation $K_+(z,z'|s)$ is

$$K_+(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\, \mathrm{e}^{\mathrm{i}(x-x')^2/4s}\, (\theta - \theta' - \mathrm{i}s\, W)^2\, I(z,z')\,. \tag{6.14}$$

This is the heat kernel corresponding to the chiral propagator used in [61].

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A. Technical lemma

This appendix is devoted to a derivation of eq. (4.22). It is in fact sufficient to prove the following relation

$$\begin{aligned}(n+1)\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_n}\mathcal{D}_B I(z,z')\Big|_{z=z'} &= n\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{n-1}}F_{A_n\,B} + \\ +\frac{1}{2}(n-1)\zeta^{A_n}T_{A_n\,B}{}^{C}\zeta^{A_{n-1}}\cdots\zeta^{A_1}&\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{n-2}}F_{A_{n-1}\,B}\,,\end{aligned} \tag{A.1}$$

with n a positive integer.

We start with an obvious identity

$$\begin{aligned}(n\,+\,1)\,\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{(A_1}\cdots\mathcal{D}_{A_n}\mathcal{D}_{B\}} &= \zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_n}\mathcal{D}_B \\ &+\sum_{i=1}^{n}(-1)^{B(A_i+\cdots+A_n)}\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{i-1}}\mathcal{D}_B\mathcal{D}_{A_i}\cdots\mathcal{D}_{A_n}\ ,\end{aligned} \tag{A.2}$$

and make use of eq. (4.17), rewritten as

$$\mathcal{D}_{(A_1}\cdots\mathcal{D}_{A_n}\mathcal{D}_{B\}}\, I(z,z')\Big|_{z=z'} = 0\,. \tag{A.3}$$

We thus have

$$\begin{aligned}0 = \zeta^{A_n}\cdots\zeta^{A_1}&\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_n}\mathcal{D}_B\, I(z,z')\Big|_{z=z'} + \\ +\sum_{i=1}^{n}(-1)^{B(A_i+\cdots+A_n)}&\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{i-1}}\mathcal{D}_B\mathcal{D}_{A_i}\cdots\mathcal{D}_{A_n}\, I(z,z')\Big|_{z=z'}\,.\end{aligned} \tag{A.4}$$

The next natural step is to represent

$$\begin{aligned}(-1)^{B(A_i+\cdots+A_n)}&\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{i-1}}\mathcal{D}_B\mathcal{D}_{A_i}\cdots\mathcal{D}_{A_n} = \\ &= -(-1)^{B(A_{i+1}+\cdots+A_n)}\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{i-1}}[\mathcal{D}_{A_i},\mathcal{D}_B\}\mathcal{D}_{A_{i+1}}\cdots\mathcal{D}_{A_n} + \\ &\quad+(-1)^{B(A_{i+1}+\cdots+A_n)}\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_i}\mathcal{D}_B\mathcal{D}_{A_{i+1}}\cdots\mathcal{D}_{A_n}\end{aligned} \tag{A.5}$$

and make use of the covariant derivative algebra (4.3), along with the observation

$$
\begin{aligned}
&(-1)^{B(A_{i+1}+\cdots+A_n)}\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{i-1}}F_{A_i\,B}\mathcal{D}_{A_{i+1}}\cdots\mathcal{D}_{A_n}\,I(z,z')\Big|_{z=z'} \\
&\quad=\begin{cases} 0\,, & i<n\,; \\ \zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{n-1}}\,F_{A_n\,B}\,, & i=n\,. \end{cases}
\end{aligned}
\tag{A.6}
$$

Repeating this procedure, each contribution to the second terms in (A.4) can be reduced to the first term plus additional terms involving graded commutators of covariant derivatives. Since the torsion $T_{AB}{}^C$ in (4.3) is constant, we then obtain

$$
\begin{aligned}
&(n+1)\,\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_n}\mathcal{D}_B I(z,z')\Big|_{z=z'}= \\
&=\sum_{i=1}^{n} i(-1)^{C(A_{i+1}+\cdots+A_n)}\zeta^{A_i}T_{A_iB}{}^{C}\zeta^{A_n}\cdots\underbrace{1}_{i}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\underbrace{\mathcal{D}_C}_{i}\cdots\mathcal{D}_{A_n}I(z,z')\Big|_{z=z'}+ \\
&\quad+n\mathrm{i}\,\zeta^{A_n}\cdots\zeta^{A_1}\mathcal{D}_{A_1}\cdots\mathcal{D}_{A_{n-1}}\,F_{A_n\,B}\,.
\end{aligned}
\tag{A.7}
$$

For the first term in the right hand side, we can again apply the previous procedure, and this now simplifies since

$$
T_{AB}{}^{C}\,[\mathcal{D}_C,\mathcal{D}_D\} = (-1)^C\,T_{AB}{}^{C}\,[\mathcal{D}_C,\mathcal{D}_D\} = \mathrm{i}\,T_{AB}{}^{C}\,F_{CD}\,. \tag{A.8}
$$

After some algebra, one then arrives at (A.1).

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